χ^2 distribution

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EECE 566

Introduction

- Let $\xi_i \sim \mathcal{N}(0, 1)$, iid.
- We are interested in $\sum_{i=1}^{\nu} \xi_i^2 = X$ (sum of ν squares of independent normal rv's)
- $X \sim \chi^2_\nu$: Chi-square with ν degrees of freedom

χ^2 distribution properties

$$
\begin{aligned}
E(\chi^2_\nu) &= E\left(\sum_{i=1}^{\nu} \xi_i^2\right) \\
&= \sum_{i=1}^{\nu} E(\xi_i^2) \\
E(\chi^2_\nu) &= \sum_{i=1}^{\nu} 1 = \nu
\end{aligned}
$$

- $E(\xi_i^2) = 1$ from problem 1.1 in assignment on statistics
- $Var(\xi_i) = E(\xi_i^2) - E(\xi_i)^2 = 1$

χ^2 distribution properties

$$
\begin{aligned}
Var(\chi^2_\nu) &= Var\left(\sum_{i=1}^{\nu} \xi_i^2\right) \\
&= \sum_{i=1}^{\nu} Var(\xi_i^2) \\
Var(\chi^2_\nu) &= \sum_{i=1}^{\nu} 2 = 2\nu
\end{aligned}
$$

- $Var(\xi_i^2) = 2$ from Problem 1.1 in assignment on statistics
- $Var(\xi_i^2) = E(\xi_i^4) - E(\xi_i^2)^2 = E(\xi_i^4) - 1^2 = 3 - 1$

χ^2 distribution properties

- Probability density function

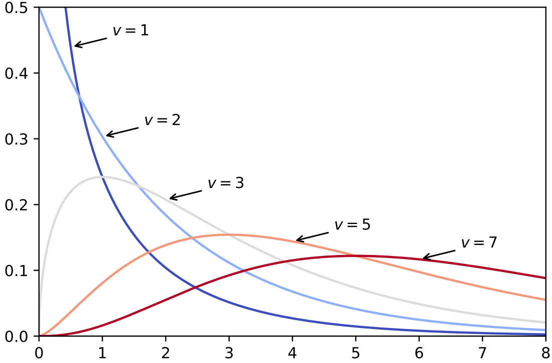
$$p(x;\,\nu) = \begin{cases} \dfrac{x^{\frac{\nu}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

- $\Gamma(x)$ is the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt,\; x > 0$$

- $\Gamma(n) = (n-1)!,\ \Gamma(1/2) = \sqrt{\pi}$
- When $\nu = 2$: Problem 1.2 in assignment on statistics

χ^2 distribution properties



χ^2 distribution properties

Matlab

- In stats toolbox: 'chi2pdf', 'chi2cdf'
- Appendix 2D: code for $Q_{\chi^2}(x)$: the right-tail probability: 'Qchipr2($\lambda = 0$)'

Python

- In scipy: 'scipy.stats.chi2' (pdf, cdf, sf, rvs, etc.)

Non-central χ^2 distribution

- Let $\xi_i \sim \mathcal{N}(\mu_i, 1)$, independent rv's
- $\sum_{i=1}^{\nu} \xi_i^2 = X \sim {\chi'_\nu}^2(\lambda)$, where $\lambda = \sum_{i=1}^{\nu} \mu_i^2$ is the non-centrality parameter
- Often arises in likelihood ratio tests

Non-central χ^2 distribution properties

$$
\begin{aligned}
E\left(\chi_\nu'^{\,2}(\lambda)\right) &= E\left(\sum_{i=1}^{\nu} \xi_i^2\right) \\
&= \sum_{i=1}^{\nu} E(\xi_i^2) \\
&= \sum_{i=1}^{\nu} E\left((\mu_i + \mathcal{N}(0,1))^2\right) \\
&= \sum_{i=1}^{\nu} \left[\mu_i^2 + 2\mu_i E(\mathcal{N}(0,1)) + E\left(\mathcal{N}(0,1)^2\right)\right] \\
E\left(\chi_\nu'^{\,2}(\lambda)\right) &= \sum_{i=1}^{\nu} [\mu_i^2 + 1] = \lambda + \nu
\end{aligned}
$$

Non-central χ^2 distribution properties

$$
\begin{aligned}
Var\left(\chi_\nu'^2(\lambda)\right) &= Var\left(\sum_{i=1}^{\nu} \xi_i^2\right) \\
&= \sum_{i=1}^{\nu} Var(\xi_i^2) \\
&= \sum_{i=1}^{\nu} Var\left((\mu_i + \mathcal{N}(0,1))^2\right) \\
&= \sum_{i=1}^{\nu} Var(\mu_i^2) + Var\left(2\mu_i \mathcal{N}(0,1)\right) + Var\left(\mathcal{N}(0,1)^2\right) \\
&\quad +2E\left((2\mu_i \mathcal{N}(0,1) - 0) \times (\mathcal{N}(0,1)^2 - 1)\right) \\
Var\left(\chi_\nu'^2(\lambda)\right) &= \sum_{i=1}^{\nu} [4\mu_i^2 + 2] = 2(2\lambda + \nu)
\end{aligned}
$$

Non-central χ^2 distribution properties

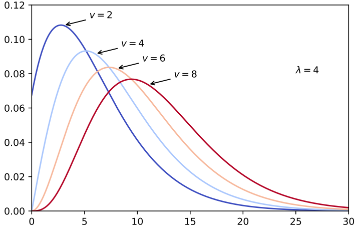
- Probability density function

$$p(x;\,\nu,\lambda) = \begin{cases} \frac{1}{2}\left(\frac{x}{\lambda}\right)^{\frac{\nu-2}{4}} e^{-\frac{x+\lambda}{2}} I_{\nu/2-1}(\sqrt{\lambda x}) & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

- $I_r(u)$: Modified Bessel function of the first kind and order r

$$I_r(u) = \sum_{k=0}^{\infty} \frac{\left(\frac{u}{2}\right)^{2k+r}}{k!\Gamma(r+k+1)}$$

Non-central χ^2 distribution properties



Non-central χ^2 distribution properties

Matlab

- In stats toolbox: ‘ncx2pdf‘, ‘ncx2pdf‘
- Appendix 2D: code for $Q_{\chi'^2}(x)$: the right-tail probability: ‘Qchipr2‘

Python

- In scipy: ‘scipy.stats.ncx2‘ (pdf, cdf, sf, rvs, etc.)