Deterministic Signals (Bayesian Approach)

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EECE 566

Introduction

- Recall detection of a known signal s_j in noise studied previously

$$\begin{aligned} H_0: &\quad x_j = \xi_j \\ H_1: &\quad x_j = s_j + \xi_j \end{aligned}$$

- We now extend to the binary case of two signals $s_j^{(0)}$ and $s_j^{(1)}$
- The detection problem is now a classification problem

$$\begin{aligned} H_0: &\quad x_j = s_j^{(0)} + \xi_j \\ H_1: &\quad x_j = s_j^{(1)} + \xi_j \end{aligned}$$

Binary deterministic signals in WGN

- $\xi_j \sim \mathcal{N}(0, \sigma^2)$, all errors have the same cost, no gains, and equal priors $\Rightarrow$ ML detector:

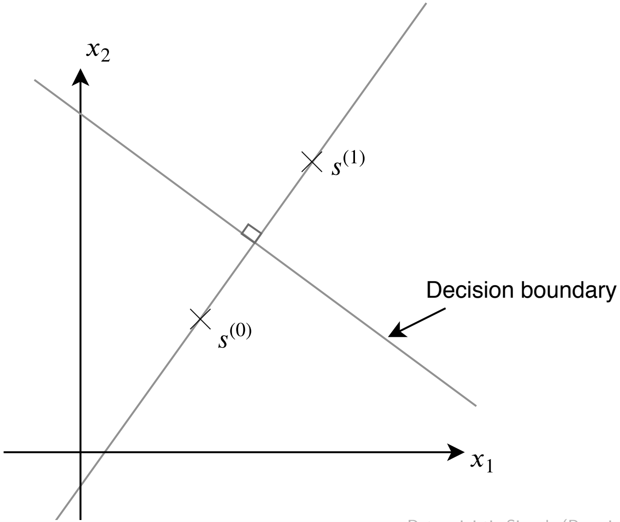
$$\frac{p(\mathbf{x}|H_1)}{p(\mathbf{x}|H_0)} \underset{1}{\overset{0}{\lessgtr}} 1$$

$$p(\mathbf{x}|H_i) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(x_j - s_j^{(i)})^2\right)$$

- Similar to the detection of multiple DC levels: minimum Euclidean distance detector $k = \text{argmin}_i D_i^2 = \text{argmin}_i \sum_{j=1}^{n}(x_j - s_j^{(i)})^2$

Binary deterministic signals in WGN (2d)

- $\mathbf{x} \in \mathbb{R}^2$, 2d case for visualization



Binary deterministic signals in WGN

… or

$$\begin{aligned} D_i^2 &= \sum_{j=1}^{n} (x_j - s_j^{(i)})^2 \\ &= \underbrace{\sum_{j=1}^{n} x_j^2}_{\text{Doesn't depend on } i} - 2\sum_{j=1}^{n} x_j s_j^{(i)} + \underbrace{\sum_{j=1}^{n} (s_j^{(i)})^2}_{\mathcal{E}^{(i)}} \\ T_i(\mathbf{x}) &= \sum_{j=1}^{n} x_j s_j^{(i)} - \frac{\mathcal{E}^{(i)}}{2} \end{aligned}$$

- The detector $k = \text{argmax}_i T_i(\mathbf{x})$ is a correlator with a bias term $-\frac{\mathcal{E}^{(i)}}{2}$ to adjust for potentially different signal energies

Binary deterministic signals in WGN

Probability of error

$$
\begin{aligned}
P_e &= P(H_1|H_0)P(H_0) + P(H_0|H_1)P(H_1) \\
&= \frac{1}{2}\left[\Pr\{T_1(\mathbf{x}) > T_0(\mathbf{x})|H_0\} + \Pr\{T_0(\mathbf{x}) > T_1(\mathbf{x})|H_1\}\right] \\
P_e &= \frac{1}{2}\left[\Pr\{T_1(\mathbf{x}) - T_0(\mathbf{x}) > 0|H_0\} + \Pr\{T_1(\mathbf{x}) - T_0(\mathbf{x}) < 0|H_1\}\right] \\
T(\mathbf{x}) &= T_1(\mathbf{x}) - T_0(\mathbf{x}), \text{ threshold is 0} \\
T(\mathbf{x}) &= \sum_{j=1}^{n} x_j(s_j^{(1)} - s_j^{(0)}) - \frac{1}{2}\mathcal{E}^{(1)} + \frac{1}{2}\mathcal{E}^{(0)}
\end{aligned}
$$

Is $T(\mathbf{x})$ Gaussian?

Binary deterministic signals in WGN

$$T(\mathbf{x}) = \sum_{j=1}^{n} x_j (s_j^{(1)} - s_j^{(0)}) - \frac{1}{2}\mathcal{E}^{(1)} + \frac{1}{2}\mathcal{E}^{(0)}$$

$T(\mathbf{x})$ is Gaussian under H_0 and H_1 because x_j's are iid Gaussian

$$\begin{aligned}
E[T(\mathbf{x})|H_0] &= \sum_{j=1}^{n} s_j^{(0)} (s_j^{(1)} - s_j^{(0)}) - \frac{1}{2}\mathcal{E}^{(1)} + \frac{1}{2}\mathcal{E}^{(0)} \\
&= \sum_{j=1}^{n} s_j^{(0)} s_j^{(1)} - \frac{1}{2}\mathcal{E}^{(1)} - \frac{1}{2}\mathcal{E}^{(0)} \\
& \sum_{j=1}^{n} (s_j^{(0)} s_j^{(1)} - \frac{1}{2} s_j^{(1)} s_j^{(1)} - \frac{1}{2} s_j^{(0)} s_j^{(0)}) \\
&= \frac{-1}{2} \sum_{j=1}^{n} (s_j^{(0)} - s_j^{(1)})^2 = \frac{-1}{2} ||\mathbf{s}^{(0)} - \mathbf{s}^{(1)}||^2
\end{aligned}$$

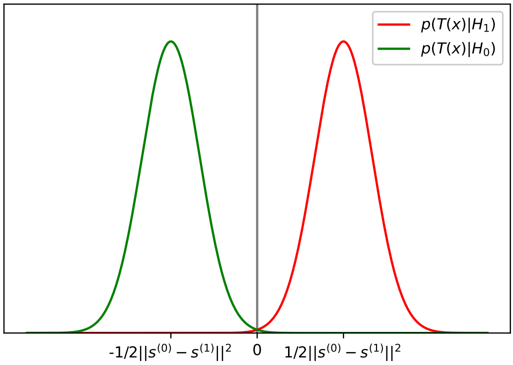
Binary deterministic signals in WGN

$$T(\mathbf{x}) \;=\; \sum_{j=1}^{n} x_j (s_j^{(1)} - s_j^{(0)}) - \frac{1}{2}\mathcal{E}^{(1)} + \frac{1}{2}\mathcal{E}^{(0)}$$

$$\begin{aligned}
\text{Similarly, } E[T(\mathbf{x})|H_1] &= \frac{1}{2}||\mathbf{s}^{(0)} - \mathbf{s}^{(1)}||^2 = -E[T(\mathbf{x})|H_0] \\
Var[T(\mathbf{x})|H_0] &= Var\left[\sum_{j=1}^{n} x_j (s_j^{(1)} - s_j^{(0)})\right] \\
&= \sum_{j=1}^{n} Var[x_j](s_j^{(1)} - s_j^{(0)})^2 \\
&= \sigma^2 ||\mathbf{s}^{(0)} - \mathbf{s}^{(1)}||^2 \\
\text{Similarly, } Var[T(\mathbf{x})|H_1] &= \sigma^2 ||\mathbf{s}^{(0)} - \mathbf{s}^{(1)}||^2 = Var[T(\mathbf{x})|H_0]
\end{aligned}$$

Binary deterministic signals in WGN

$$T(\mathbf{x}) \;=\; \sum_{j=1}^{n} x_j (s_j^{(1)} - s_j^{(0)}) - \frac{1}{2}\mathcal{E}^{(1)} + \frac{1}{2}\mathcal{E}^{(0)}$$



Binary deterministic signals in WGN

- Mean shifted Gauss-Gauss problem:

$$\begin{aligned} P_e &= \frac{1}{2}(\Pr\{T(\mathbf{x}) > 0 | H_0\} + \Pr\{T(\mathbf{x}) < 0 | H_1\}) \\ &= \Pr\{T(\mathbf{x}) > 0 | H_0\} \\ P_e &= Q\left(\frac{\frac{1}{2}||\mathbf{s}^{(0)} - \mathbf{s}^{(1)}||^2}{\sigma||\mathbf{s}^{(0)} - \mathbf{s}^{(1)}||}\right) = Q\left(\sqrt{d^2}\right) \\ d^2 &= \frac{||\mathbf{s}^{(0)} - \mathbf{s}^{(1)}||^2}{4\sigma^2} \quad \text{deflection coefficient} \end{aligned}$$

Binary deterministic signals in WGN: optimal signal design

- **Problem:** How to design $\mathbf{s}^{(0)}, \mathbf{s}^{(1)}$ under energy constraint: $\frac{1}{2}(\mathcal{E}^{(1)} + \mathcal{E}^{(0)}) = \overline{\mathcal{E}} = const.$ to minimize P_e?

$$\begin{aligned}
||\mathbf{s}^{(0)} - \mathbf{s}^{(1)}||^2 &= (\mathbf{s}^{(0)} - \mathbf{s}^{(1)})^T \cdot (\mathbf{s}^{(0)} - \mathbf{s}^{(1)}) \\
&= \mathbf{s}^{(0)T} \cdot \mathbf{s}^{(0)} + \mathbf{s}^{(1)T} \cdot \mathbf{s}^{(1)} - 2\mathbf{s}^{(1)T} \cdot \mathbf{s}^{(0)} \\
&= 2\overline{\mathcal{E}} - 2\mathbf{s}^{(1)T} \cdot \mathbf{s}^{(0)} \\
&= 2\overline{\mathcal{E}} \left(1 - \underbrace{\frac{\mathbf{s}^{(1)T} \cdot \mathbf{s}^{(0)}}{\frac{1}{2}[||\mathbf{s}^{(0)}||^2 + ||\mathbf{s}^{(1)}||^2]}}_{\rho_S} \right)
\end{aligned}$$

- $\rho_S = \frac{\mathbf{s}^{(1)T} \cdot \mathbf{s}^{(0)}}{1/2[||\mathbf{s}^{(0)}||^2 + ||\mathbf{s}^{(1)}||^2]}$ = signal correlation coefficient

Binary deterministic signals in WGN: optimal signal design

- $\rho_S = \frac{\mathbf{s}^{(1)T} \cdot \mathbf{s}^{(0)}}{1/2[||\mathbf{s}^{(0)}||^2 + ||\mathbf{s}^{(1)}||^2]}$ = signal correlation coefficient
- $-1 \leq \rho_S \leq 1$ because

$$\rho_S = \underbrace{\frac{\mathbf{s}^{(1)T} \cdot \mathbf{s}^{(0)}}{||\mathbf{s}^{(0)}|| \times ||\mathbf{s}^{(1)}||}}_{|.| \leq 1 \text{CS Ineq}} \times \underbrace{\frac{||\mathbf{s}^{(0)}|| \times ||\mathbf{s}^{(1)}||}{\frac{1}{2}[||\mathbf{s}^{(0)}||^2 + ||\mathbf{s}^{(1)}||^2]}}_{\leq 1 \text{AG Ineq}}$$

- CS Ineq: Cauchy–Schwarz inequality
- AG Ineq: Inequality of arithmetic and geometric means $\frac{ab}{1/2(a^2+b^2)} \leq 1, \forall a, b, a^2 + b^2 \neq 0$

Binary deterministic signals in WGN: optimal signal design

$$
\begin{aligned}
P_e &= Q\left(\sqrt{\frac{||\mathbf{s}^{(0)} - \mathbf{s}^{(1)}||^2}{4\sigma^2}}\right) \\
&= Q\left(\sqrt{\frac{2\overline{\mathcal{E}}(1-\rho_S)}{4\sigma^2}}\right)
\end{aligned}
$$

will be minimum if $\rho_S = -1$, i. e., anticorrelated signals

$$
\rho_S = -1 \Leftrightarrow \begin{cases} ||\mathbf{s}^{(0)}|| = ||\mathbf{s}^{(1)}|| & \text{(from AG ineq)} \\ \text{and} & \\ \mathbf{s}^{(0)} = -\mathbf{s}^{(1)} & \text{(from CS ineq)} \end{cases}
$$

For this optimal design: $P_e = Q\left(\sqrt{\overline{\mathcal{E}}/\sigma^2}\right)$