Bayesian Hypothesis Testing (binary)

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EECE 566

Introduction

- Prior probabilities $P(H_0), P(H_1)$ available
  - Disease detection with a good knowledge of infection rates
  - Artificial bit stream sent with known 0's and 1's probabilities
  - Not applicable to camera ID using fingerprints
- Cost assigned to each decision–hypothesis pair: $C_{00}, C_{11}, C_{01}, C_{11}$

		Hypothesis 0	Hypothesis 1
Decision	0	C_{00}	C_{01}
	1	C_{10}	C_{11}

- $C_{00}, C_{11} \leq 0$ **gains** of true negatives and true positives, and $C_{01}, C_{10} \geq 0$ **costs** of missed detection and false alarm

Bayes risk minimization

- $C = \sum_{i,j=0}^{1} C_{ij} P(H_i|H_j) P(H_j)$ Bayes Risk is the expected cost/gain. Recall from previous lecture (Appendix 3b), the minimum Bayes Risk detector:

$$
\begin{aligned}
L(\mathbf{x}) = \frac{p(\mathbf{x}|H_1)}{p(\mathbf{x}|H_0)} &\underset{1}{\overset{0}{\lessgtr}} \frac{C_{10} - C_{00}}{C_{01} - C_{11}} \frac{P(H_0)}{P(H_1)} = \gamma \\
\frac{p(\mathbf{x}|H_1)P(H_1)}{p(\mathbf{x}|H_0)P(H_0)} &\underset{1}{\overset{0}{\lessgtr}} \frac{C_{10} - C_{00}}{C_{01} - C_{11}} \\
\text{(Bayes rule)}\ \frac{p(H_1|\mathbf{x})}{p(H_0|\mathbf{x})} &\underset{1}{\overset{0}{\lessgtr}} \frac{C_{10} - C_{00}}{C_{01} - C_{11}}
\end{aligned}
$$

- Bayes rule $P(A \wedge B) = P(A|B)P(B) = P(B|A)P(B)$

Bayes risk minimization, equal costs, zero gains

- Equal costs $C_{01} = C_{10}$, zero gains $C_{00}, C_{11} = 0$:

$$p(H_1|\mathbf{x}) \underset{1}{\overset{0}{\lessgtr}} p(H_0|\mathbf{x}) \qquad \text{MAP detector}$$

- MAP $=$ Maximum a posteriori detector
- Bayes risk for $C_{01} = C_{10}$, and $C_{00} = C_{11} = 0$: Total probability of error

$$P_e = P(H_1|H_0)P(H_0) + P(H_0|H_1)P(H_1)$$

Bayes risk minimization, equal costs, zero gains, equal priors

- Equal costs $C_{01} = C_{10}$, zero gains $C_{00}, C_{11} = 0$, and equal priors $P(H_0) = P(H_1)$

$$p(H_1|\mathbf{x}) \underset{1}{\overset{0}{\lessgtr}} p(H_0|\mathbf{x})$$

$$\text{(Bayes rule)}\ \frac{p(\mathbf{x}|H_1)}{p(\mathbf{x}|H_0)} \underset{1}{\overset{0}{\lessgtr}} 1 \quad \text{ML detector}$$

- ML = Maximum Likelihood

Example: Detection of DC level in WGN

Equal costs, no gains, equal priors $\Rightarrow$ ML detector decides H_1 when $p(\mathbf{x}|H_1)/p(\mathbf{x}|H_0) > 1$

$$
\begin{aligned}
H_0 : &\quad x_i = \xi_i \\
H_1 : &\quad x_i = A + \xi_i
\end{aligned}
$$

A DC Signal, ξ_i iid Gaussian noise, $\xi_i \sim \mathcal{N}(0, \sigma^2)$
$p(\mathbf{x}|H_0) = \mathcal{N}(0, \sigma^2)$ and $p(\mathbf{x}|H_1) = \mathcal{N}(A, \sigma^2)$

$$
\begin{aligned}
\log L(\mathbf{x}) = \frac{-1}{2\sigma^2} \sum_{i=1}^{n} A^2 - 2x_i A &\quad > \quad \log 1 \\
\frac{1}{2\sigma^2}(2n\overline{\mathbf{x}}A - nA^2) &\quad > \quad 0 \\
\overline{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} x_i &\quad > \quad \frac{A}{2}
\end{aligned}
$$

Example: Detection of DC level in WGN

$$\bar{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad > \quad A/2$$

$$\bar{\mathbf{x}} \quad \sim \quad \begin{cases} \mathcal{N}(0, \sigma^2/n) & \text{under } H_0 \\ \mathcal{N}(A, \sigma^2/n) & \text{under } H_1 \end{cases}$$

Probability of error:

$$\begin{aligned} P_e &= P(H_0|H_1)P(H_1) + P(H_1|H_0)P(H_0) \\ &= \frac{1}{2}[P(H_0|H_1) + P(H_1|H_0)] \\ &= \frac{1}{2}\left(Q\left(\frac{A/2 - 0}{\sqrt{\sigma^2/n}}\right) + Q\left(\frac{A/2}{\sqrt{\sigma^2/n}}\right)\right) \\ P_e &= Q\left(\sqrt{\frac{nA^2}{4\sigma^2}}\right) = Q(d^2) \end{aligned}$$

Example: Detection of DC level in WGN