Bayesian Hypothesis Testing (binary)

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EECE 566

Introduction

- Prior probabilities $P(H_0), P(H_1)$ available
 - Disease detection with a good knowledge of infection rates
 - Artificial bit stream sent with known 0's and 1's probabilities
 - Not applicable to camera ID using fingerprints
- Cost assigned to each decision–hypothesis pair: $C_{00}, C_{11}, C_{01}, C_{11}$

$$\begin{array}{c|c} & \mathsf{Hypothesis} \\ 0 & 1 \\ \hline \mathsf{Decision} & \begin{matrix} \mathsf{0} \\ 1 \end{matrix} \begin{vmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{vmatrix}$$

• $C_{00}, C_{11} \leq 0$ gains of true negatives and true positives, and $C_{01}, C_{10} \geq 0$ costs of missed detection and false alarm

Bayes risk minimization

• $C = \sum_{i,j=0}^{1} C_{ij} P(H_i|H_j) P(H_j)$ Bayes Risk is the expected cost/gain. Recall from previous lecture (Appendix 3b), the minimum Bayes Risk detector:

$$\begin{split} L(\mathbf{x}) &= \frac{p(\mathbf{x}|H_1)}{p(\mathbf{x}|H_0)} \quad \stackrel{0}{\leq} \quad \frac{C_{10} - C_{00}}{C_{01} - C_{11}} \frac{P(H_0)}{P(H_1)} = \gamma \\ &\frac{p(\mathbf{x}|H_1)P(H_1)}{p(\mathbf{x}|H_0)P(H_0)} \quad \stackrel{0}{\leq} \quad \frac{C_{10} - C_{00}}{C_{01} - C_{11}} \\ \\ \mathsf{Bayes rule}) \quad \frac{p(H_1|\mathbf{x})}{p(H_0|\mathbf{x})} \quad \stackrel{0}{\leq} \quad \frac{C_{10} - C_{00}}{C_{01} - C_{11}} \end{split}$$

• Bayes rule $P(A \wedge B) = P(A|B)P(B) = P(B|A)P(B)$

Bayes risk minimization, equal costs, zero gains

• Equal costs $C_{01} = C_{10}$, zero gains $C_{00}, C_{11} = 0$:

$$p(H_1|\mathbf{x}) \stackrel{0}{\underset{1}{\leq}} p(H_0|\mathbf{x})$$
 MAP detector

- MAP = Maximum a posteriori detector
- Bayes risk for $C_{01} = C_{10}$, and $C_{00} = C_{11} = 0$: Total probability of error

$$P_e = P(H_1|H_0)P(H_0) + P(H_0|H_1)P(H_1)$$

Bayes risk minimization, equal costs, zero gains, equal priors

• Equal costs $C_{01} = C_{10}$, zero gains $C_{00}, C_{11} = 0$, and equal priors $P(H_0) = P(H_1)$

• ML = Maximum Likelihood

Example: Detection of DC level in WGN

Equal costs, no gains, equal priors \Rightarrow ML detector decides H_1 when $p(\mathbf{x}|H_1)/p(\mathbf{x}|H_0) > 1$

$$H_0: \quad x_i = \xi_i$$
$$H_1: \quad x_i = A + \xi_i$$

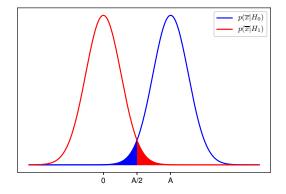
A DC Signal, ξ_i iid Gaussian noise, $\xi_i \sim \mathcal{N}(0, \sigma^2)$ $p(\mathbf{x}|H_0) = \mathcal{N}(0, \sigma^2)$ and $p(\mathbf{x}|H_1) = \mathcal{N}(A, \sigma^2)$

$$\log L(\mathbf{x}) = \frac{-1}{2\sigma^2} \sum_{i=1}^n A^2 - 2x_i A > \log 1$$
$$\frac{1}{2\sigma^2} (2n\overline{\mathbf{x}}A - nA^2) > 0$$
$$\overline{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^n x_i > \frac{A}{2}$$

Example: Detection of DC level in WGN

$$\begin{split} \overline{\mathbf{x}} &= \frac{1}{n} \sum_{i=1}^{n} x_i > A/2 \\ \overline{\mathbf{x}} \sim \begin{cases} \mathcal{N}(0, \sigma^2/n) & \text{under } H_0 \\ \mathcal{N}(A, \sigma^2/n) & \text{under } H_1 \end{cases} \\ \text{Probability of error: } P_e &= P(H_0|H_1)P(H_1) + P(H_1|H_0)P(H_0) \\ &= \frac{1}{2} [P(H_0|H_1) + P(H_1|H_0)] \\ &= \frac{1}{2} \left(Q\left(\frac{A/2 - 0}{\sqrt{\sigma^2/n}}\right) + Q\left(\frac{A/2}{\sqrt{\sigma^2/n}}\right) \right) \\ P_e &= Q\left(\sqrt{\frac{nA^2}{4\sigma^2}}\right) = Q(d^2) \end{split}$$

Example: Detection of DC level in WGN



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