Composite Hypothesis Testing

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EECE 566

Introduction

- **Simple Hypothesis Testing** $=$ Complete knowledge of pdf's under H_0 and H_1
- **Composite Hypothesis Testing** $=$ pdf's under H_0 and/or H_1 have unknown parameters
  - usually the case in practice
  - unknown noise variance
  - unknown signal modulation / attenuation / transformation

Example: Detection of unknown DC level in WGN

$$H_0: \quad x_i = \xi_i$$

$$H_1: \quad x_i = A + \xi_i$$

$\xi_i \sim \mathcal{N}(0, \sigma^2)$, iid with known σ^2

A is unknown. Assume $A > 0$

$$\text{LRT} = \frac{p(\mathbf{x}; A, H_1)}{p(\mathbf{x}; H_0)} = \frac{(2\pi\sigma^2)^{-n/2} \exp\left(\frac{-1}{2\sigma^2} \sum_{i=1}^n (x_i - A)^2\right)}{(2\pi\sigma^2)^{-n/2} \exp\left(\frac{-1}{2\sigma^2} \sum_{i=1}^n x_i^2\right)}$$

$$\overset{0}{\underset{1}{\lessgtr}} \quad \gamma$$

$$\frac{-1}{2\sigma^2}\left(-2A\sum_{i=1}^n x_i + nA^2\right) \overset{0}{\underset{1}{\lessgtr}} \ln\gamma$$

Example: Detection of unknown DC level in WGN

$$\frac{-1}{2\sigma^2}\left(-2A\sum_{i=1}^{n} x_i + nA^2\right) \underset{1}{\overset{0}{\lessgtr}} \ln\gamma$$

$$A\sum_{i=1}^{n} x_i \underset{1}{\overset{0}{\lessgtr}} \sigma^2 \ln\gamma + \frac{nA^2}{2}$$

$$\overline{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} x_i \underset{1}{\overset{0}{\lessgtr}} \frac{\sigma^2 \ln\gamma}{nA} + \frac{A}{2} = \gamma',\ (A > 0)$$

- γ' depends on unknown parameter A, but this dependence is only illusory

Example: Detection of unknown DC level in WGN

$$\overline{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \underset{1}{\overset{0}{\lessgtr}} \quad \frac{\sigma^2 \ln \gamma}{nA} + \frac{A}{2} = \gamma', \ (A > 0)$$

- Under H_0: $\overline{\mathbf{x}} \sim \mathcal{N}(0, \sigma^2/n)$

$$\begin{aligned} P_{\mathsf{FA}} &= \mathsf{Pr}\{\overline{\mathbf{x}} > \gamma' | H_0\} \\ &= Q\left(\frac{\gamma'}{\sqrt{\sigma^2/n}}\right) \\ \Rightarrow \gamma' &= \sqrt{\sigma^2/n}\, Q^{-1}(P_{\mathsf{FA}}) \end{aligned}$$

- We can set γ' to get NP detector for given P_{FA} without knowing A!

Example: Detection of unknown DC level in WGN

- Under H_1: $\overline{\mathbf{x}} \sim \mathcal{N}(A, \sigma^2/n)$

$$P_{\mathsf{D}} = Q\left(\frac{\gamma' - A}{\sqrt{\sigma^2/n}}\right) = Q\left(Q^{-1}(P_{\mathsf{FA}}) - \sqrt{\frac{nA^2}{\sigma^2}}\right)$$

- P_{D} does depend on A
- But NP detector: $\overline{\mathbf{x}} \underset{1}{\overset{0}{\lessgtr}} \sqrt{\sigma^2/n}\, Q^{-1}(P_{\mathsf{FA}})$ yields the highest P_{D} for a given P_{FA} **independently of** A (as long as $A > 0$)
- UMP: Universally Most Powerful test

Example: Detection of unknown DC level in WGN

$$H_0: \quad x_i = \xi_i$$
$$H_1: \quad x_i = A + \xi_i$$

$\xi_i \sim \mathcal{N}(0, \sigma^2)$, iid with known σ^2

A is unknown. ~~Assume $A > 0$~~

- if $A > 0$: $\overline{\mathbf{x}} \underset{1}{\overset{0}{\lessgtr}} \sqrt{\sigma^2/n}\, Q^{-1}(P_{\mathsf{FA}})$
- if $A < 0$: $\overline{\mathbf{x}} \underset{0}{\overset{1}{\lessgtr}} -\sqrt{\sigma^2/n}\, Q^{-1}(P_{\mathsf{FA}})$ (same steps as for $A > 0$)
- UMP does not exist

Example: Detection of unknown DC level in WGN

Summary

$$H_0 : \quad A = 0$$
$$H_1 : \quad A > 0$$

(or $A < 0$) one-sided test: UMP exists

$$H_0 : \quad A = 0$$
$$H_1 : \quad A \neq 0$$

Two-sided test: UMP does not exist

When UMP does not exist: use a suboptimal detector and compare to the best possible detector: the one assuming **known A** (the so-called **Clairvoyant detector**)

Example: Detection of unknown DC level in WGN

Clairvoyant detector decides:

$$\bar{\mathbf{x}} \underset{1}{\overset{0}{\lessgtr}} \gamma, \text{ for } A > 0$$

$$\bar{\mathbf{x}} \underset{0}{\overset{1}{\lessgtr}} -\gamma, \text{ for } A < 0$$

where $\gamma = \sqrt{\sigma^2/n}\, Q^{-1}(P_{\mathsf{FA}})$.

This detector is unrealizable because the choice of the NP test depends on A

$$P_{\mathsf{D}} = Q\left(\frac{\gamma - A}{\sqrt{\sigma^2/n}}\right) = Q\left(Q^{-1}(P_{\mathsf{FA}}) - \sqrt{\frac{nA^2}{\sigma^2}}\right), \text{ for } A > 0$$

$$P_{\mathsf{D}} = 1 - Q\left(\frac{-\gamma - A}{\sqrt{\sigma^2/n}}\right) = Q\left(Q^{-1}(P_{\mathsf{FA}}) - \sqrt{\frac{nA^2}{\sigma^2}}\right), \text{ for } A < 0$$

Example: Detection of unknown DC level in WGN

Suboptimal detector decides

$$|\overline{\mathbf{x}}| \underset{1}{\overset{0}{\lessgtr}} \gamma, \text{ for all } A$$

Under H_0: $\overline{\mathbf{x}} \sim \mathcal{N}(0, \sigma^2/n)$

$$\begin{aligned} P_{\mathsf{FA}} &= \Pr\{|\overline{\mathbf{x}}| > \gamma | H_0\} \\ &= 2\Pr\{\overline{\mathbf{x}} > \gamma | H_0\} \\ &= 2Q\left(\frac{\gamma}{\sqrt{\sigma^2/n}}\right) \\ \Rightarrow \gamma &= \sqrt{\sigma^2/n}\, Q^{-1}(P_{\mathsf{FA}}/2) \end{aligned}$$

Example: Detection of unknown DC level in WGN

Suboptimal detector decides:

$$|\overline{\mathbf{x}}| \underset{1}{\overset{0}{\lessgtr}} \gamma, \text{ for all } A$$

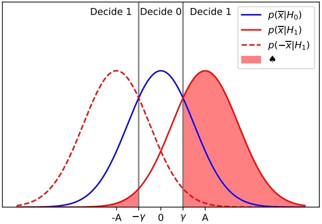

figure for $A > 0$

Example: Detection of unknown DC level in WGN

Suboptimal detector decides:

$$|\bar{\mathbf{x}}| \underset{1}{\overset{0}{\lessgtr}} \gamma, \text{ for all } A$$

$$\begin{aligned} P_{\mathsf{D}} &= \overbrace{\left(Q\left(\frac{\gamma - A}{\sqrt{\sigma^2/n}}\right) + Q\left(\frac{\gamma-(-A)}{\sqrt{\sigma^2/n}}\right)\right)}^{\spadesuit} p(A>0) \\ &\quad + \spadesuit \cdot p(A<0) \\ &= \spadesuit \\ P_{\mathsf{D}} &= Q\left(Q^{-1}(P_{\mathsf{FA}}/2) - \sqrt{\frac{nA^2}{\sigma^2}}\right) + Q\left(Q^{-1}(P_{\mathsf{FA}}/2) + \sqrt{\frac{nA^2}{\sigma^2}}\right) \end{aligned}$$

Example: Detection of unknown DC level in WGN

$$\sigma^2 = 1,\ n = 10,\ P_{\mathsf{FA}} = 0.1$$