Composite Hypothesis Testing Approaches

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Introduction

$$\begin{aligned} H_0 : &\quad \mathbf{x} \sim p(\mathbf{x}; \boldsymbol{\theta}_0) \\ H_1 : &\quad \mathbf{x} \sim p(\mathbf{x}; \boldsymbol{\theta}_1) \end{aligned}$$

- $\boldsymbol{\theta}_0$ (resp. $\boldsymbol{\theta}_1$) is the vector parameter under H_0 (resp. under H_1)
- $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$ may be different parameters

Bayesian approach

- Assign prior pdf's to the parameters under both H_0 and H_1: $p(\boldsymbol{\theta}_0)$ and $p(\boldsymbol{\theta}_1)$
- Convert to simple HT by finding

$$
\begin{aligned}
p(\mathbf{x}|H_0) &= \int p(\mathbf{x}|\boldsymbol{\theta}_0, H_0)p(\boldsymbol{\theta}_0)\mathrm{d}\boldsymbol{\theta}_0 \\
p(\mathbf{x}|H_1) &= \int p(\mathbf{x}|\boldsymbol{\theta}_1, H_1)p(\boldsymbol{\theta}_1)\mathrm{d}\boldsymbol{\theta}_1
\end{aligned}
$$

$$
\text{(LRT):}\ \frac{p(\mathbf{x}|H_1)}{p(\mathbf{x}|H_0)} = \frac{\int p(\mathbf{x}|\boldsymbol{\theta}_1, H_1)p(\boldsymbol{\theta}_1)\mathrm{d}\boldsymbol{\theta}_1}{\int p(\mathbf{x}|\boldsymbol{\theta}_0, H_0)p(\boldsymbol{\theta}_0)\mathrm{d}\boldsymbol{\theta}_0}
$$

Bayesian approach

- $p(\boldsymbol{\theta}_0)$ and $p(\boldsymbol{\theta}_1)$ may be difficult to assign
- In absence of knowledge, use non-informative priors, e. g. uniform pdf
- **Unknown DC level problem**
  - Range of $\boldsymbol{\theta}_1 = A$ is infinite $\implies$ cannot be uniform on $\mathbb{R}$
  - We can choose $A \sim \mathcal{N}(0, \sigma_A^2)$ and let $\sigma_A^2 \to \infty$

Example: Detection of **unknown** DC level in WGN - Bayesian Approach

$$H_0: \quad x_i = \xi_i$$
$$H_1: \quad x_i = A + \xi_i$$

$\xi_i \sim \mathcal{N}(0, \sigma^2)$, with known σ^2

A is unknown. $A \sim \mathcal{N}(0, \sigma_A^2)$

(LRT): $\dfrac{p(\mathbf{x}|H_1)}{p(\mathbf{x}|H_0)} \quad = \quad \dfrac{\int p(\mathbf{x}|A, H_1)p(A)\mathrm{dA}}{p(\mathbf{x}|H_0)} \underset{1}{\overset{0}{\lessgtr}} \gamma$

Example: Detection of **unknown** DC level in WGN (Bayesian Approach)

$$
\begin{aligned}
p(\mathbf{x}|H_1) &= \int_{-\infty}^{+\infty} (2\pi\sigma^2)^{-n/2} \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i - A)^2\right) \\
&\quad \cdot (2\pi\sigma_A^2)^{-1/2} \exp\left(\frac{-A^2}{2\sigma_A^2}\right) \mathrm{dA} \\
&= (2\pi\sigma^2)^{-n/2}(2\pi\sigma_A^2)^{-1/2} \\
&\quad \cdot \int_{-\infty}^{+\infty} \exp \underbrace{\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i - A)^2 - \frac{A^2}{2\sigma_A^2}\right)}_{-\frac{1}{2}F(A) = -\frac{1}{2}\left[\frac{1}{\triangle^2}(A-\heartsuit)^2 + \diamondsuit\right]} \mathrm{dA}
\end{aligned}
$$

Example: Detection of unknown DC level in WGN (Bayesian Approach)

$$
\begin{aligned}
F(A) &= \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2 - \frac{2An}{\sigma^2}\overline{\mathbf{x}} + \frac{nA^2}{\sigma^2} + \frac{A^2}{\sigma_A^2} \\
&= \underbrace{\left(\frac{n}{\sigma^2} + \frac{1}{\sigma_A^2}\right)}_{1/\sigma^2_{A|\mathbf{x}}} A^2 - \frac{2An}{\sigma^2}\overline{\mathbf{x}} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2 \\
&= \frac{A^2}{\sigma^2_{A|\mathbf{x}}} - \frac{2An\sigma^2_{A|\mathbf{x}}}{\sigma^2\sigma^2_{A|\mathbf{x}}}\overline{\mathbf{x}} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2 \\
F(A) &= \frac{1}{\sigma^2_{A|\mathbf{x}}}\left(A - \frac{n\sigma^2_{A|\mathbf{x}}\overline{\mathbf{x}}}{\sigma^2}\right)^2 - \frac{n^2\overline{\mathbf{x}}^2\sigma^2_{A|\mathbf{x}}}{\sigma^4} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2
\end{aligned}
$$

Example: Detection of **unknown** DC level in WGN (Bayesian Approach)

$$
\begin{aligned}
F(A) &= \frac{1}{\sigma^2_{A/\mathbf{x}}}\left(A - \frac{n\sigma^2_{A|\mathbf{x}}\overline{\mathbf{x}}}{\sigma^2}\right)^2 - \frac{n^2\overline{\mathbf{x}}^2\sigma^2_{A|\mathbf{x}}}{\sigma^4} \\
&\quad + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2 \\
\int \exp(-\frac{1}{2}F(A))\mathrm{dA} &= \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{n} x_i^2\right)\exp\left(\frac{n^2\overline{\mathbf{x}}^2\sigma^2_{A|\mathbf{x}}}{2\sigma^4}\right) \\
&\quad \underbrace{\cdot\int_{-\infty}^{+\infty}\exp\left(\frac{-1}{2\sigma^2_{A|\mathbf{x}}}\left(A - \frac{n\sigma^2_{A|\mathbf{x}}\overline{\mathbf{x}}}{\sigma^2}\right)^2\right)\mathrm{dA}}_{\text{Gaussian integral } = \sqrt{2\pi\sigma^2_{A|\mathbf{x}}}}
\end{aligned}
$$

Example: Detection of unknown DC level in WGN (Bayesian Approach)

$$\frac{p(\mathbf{x}|H_1)}{p(\mathbf{x}|H_0)} = \frac{\cancel{(2\pi\sigma^2)^{-n/2}}(2\pi\sigma_A^2)^{-1/2}\cancel{\exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{n} x_i^2\right)}}{\cancel{(2\pi\sigma^2)^{-n/2}}\cancel{\exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{n} x_i^2\right)}}$$

$$\cdot \exp\left(\frac{n^2\overline{\mathbf{x}}^2\sigma_{A|\mathbf{x}}^2}{2\sigma^4}\right)\sqrt{2\pi\sigma_{A|\mathbf{x}}^2}$$

$$\frac{p(\mathbf{x}|H_1)}{p(\mathbf{x}|H_0)} = \left(\frac{2\pi\sigma_{A|\mathbf{x}}^2}{2\pi\sigma_A^2}\right)^{1/2} \exp\left(\frac{n^2\overline{\mathbf{x}}^2\sigma_{A|\mathbf{x}}^2}{2\sigma^4}\right)$$

Example: Detection of unknown DC level in WGN (Bayesian Approach)

$$\frac{p(\mathbf{x}|H_1)}{p(\mathbf{x}|H_0)} = \left(\frac{2\pi\sigma^2_{A|\mathbf{x}}}{2\pi\sigma^2_A}\right)^{1/2} \exp\left(\frac{n^2\overline{\mathbf{x}}^2\sigma^2_{A|\mathbf{x}}}{2\sigma^4}\right)$$

Taking logarithms, putting data independent terms to the r.h.s.

$$\frac{n^2\sigma^2_{A|\mathbf{x}}}{2\sigma^4}\overline{\mathbf{x}}^2 \underset{1}{\overset{0}{\lessgtr}} \ln\gamma\frac{\sigma_{A|\mathbf{x}}}{\sigma^2_A}$$

$$\overline{\mathbf{x}}^2 \underset{1}{\overset{0}{\lessgtr}} \gamma'$$

$$|\overline{\mathbf{x}}| \underset{1}{\overset{0}{\lessgtr}} \sqrt{\gamma'}$$

Example: Detection of unknown DC level in WGN (Bayesian Approach)

$$
\begin{aligned}
H_0: &\quad x_i = \xi_i & \overline{\mathbf{x}} \sim \mathcal{N}(0, \sigma^2/n) \\
H_1: &\quad x_i = A + \xi_i & \overline{\mathbf{x}} \sim \mathcal{N}(0, \sigma_A^2 + \sigma^2/n)
\end{aligned}
$$

$$
\begin{aligned}
|\overline{\mathbf{x}}| &\underset{1}{\overset{0}{\lessgtr}} \gamma \\
P_{\mathsf{FA}} &= \mathsf{Pr}\{|\overline{\mathbf{x}}| > \gamma | H_0\} = 2 \cdot \mathsf{Pr}\{\overline{\mathbf{x}} > \gamma | H_0\} \\
P_{\mathsf{FA}} &= 2Q\left(\frac{\gamma}{\sqrt{\sigma^2/n}}\right) \\
\gamma &= \sqrt{\sigma^2/n}\, Q^{-1}(P_{\mathsf{FA}}/2)
\end{aligned}
$$

- We do not need σ_A^2 to set a threshold for a given P_{FA}
- But P_{D} will require the knowledge of σ_A^2

Example: Detection of unknown DC level in WGN (Bayesian Approach)

$$
\begin{aligned}
H_0: &\quad x_i = \xi_i & \overline{\mathbf{x}} \sim \mathcal{N}(0, \sigma^2/n) \\
H_1: &\quad x_i = A + \xi_i & \overline{\mathbf{x}} \sim \mathcal{N}(0, \sigma_A^2 + \sigma^2/n)
\end{aligned}
$$

$$
\begin{aligned}
|\overline{\mathbf{x}}| &\underset{1}{\overset{0}{\lessgtr}} \gamma \\
P_{\mathsf{D}} &= \mathsf{Pr}\{|\overline{\mathbf{x}}| > \gamma | H_1\} = 2 \cdot \mathsf{Pr}\{\overline{\mathbf{x}} > \gamma | H_1\} \\
P_{\mathsf{D}} &= 2Q\left(\frac{\gamma}{\sqrt{\sigma_A^2 + \sigma^2/n}}\right) \\
P_{\mathsf{D}} &= 2Q\left(\frac{\sqrt{\sigma^2/n}\,Q^{-1}(P_{\mathsf{FA}}/2)}{\sqrt{\sigma_A^2 + \sigma^2/n}}\right)
\end{aligned}
$$

Example: Detection of **unknown** DC level in WGN (Bayesian Approach)

$$
\begin{aligned}
H_0 : \quad & x_i = \xi_i & \overline{\mathbf{x}} \sim \mathcal{N}(0, \sigma^2/n) \\
H_1 : \quad & x_i = A + \xi_i & \overline{\mathbf{x}} \sim \mathcal{N}(0, \sigma_A^2 + \sigma^2/n)
\end{aligned}
$$

$$
P_{\mathsf{D}} \quad = \quad 2Q\left(\frac{\sqrt{\sigma^2/n}Q^{-1}(P_{\mathsf{FA}}/2)}{\sqrt{\sigma_A^2 + \sigma^2/n}}\right)
$$

- $\sigma_A^2 = 0 \Rightarrow P_{\mathsf{D}} = P_{\mathsf{FA}}$: Random guesser because $H_0 = H_1$
- $\sigma_A^2 \to \infty \Rightarrow P_{\mathsf{D}} \to 1$: Perfect detection. A will be large enough with high probability

Example: Detection of **unknown** DC level in WGN - Bayesian Approach

$$n = 10,\ \sigma^2 = 1$$